A NOTE ON THE DECOMPOSITION OF TREES INTO ISOMORPHIC SUBTREES

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Abstract

Caro and Schonheim gave a necessary condition for a tree H to be the union of pairwise edge disjoint subtrees, each isomorphic to a given tree G, and showed that this condition is not sufficient in general. They raised the following question:

Does the sufficiency of their condition for a given tree G and for all trees H of the same size as G imply its sufficiency for the given tree G and for all trees H?

Here we answer this question affirmatively.

Our notation is similar to that of Caro and Schonheim in [1]. All graphs considered in this note are finite. A graph H is said to have a G-decomposition if it is the union of pairwise edge-disjoint subgraphs, each isomorphic to G. We denote this situation by $G|H$.

We denote by $E(G)$ the set of edges of G and put $e(G) = |E(G)|$.

If u is a cut point of a tree T and $\{(u,z_i): 1 \le i \le s\}$ is the set of edges incident with u, then $T-u$ is a forest consisting of trees $T_1,\dots,T_s$ where $z_i \in T_i$ for $1 \le i \le s$. The branches of T at u are the trees $T_i \cup (u,z_i)$, $1 \le i \le s$. If v is a vertex of T, $v \ne u$, let $B(u,v,T)$ denote the unique branch of T at u that contains v. We denote by $\mathcal{B}(u,T)$ the set of all branches at u.

Denote by $d_{t,k}(u,T)$ the number of branches B at u such that $e(B) \equiv t \pmod k$; the (mod k) branching vector of u is the vector $d_k(u,T) = (d_{1,k}(u,T), d_{2,k}(u,T), \dots, d_{k-1,k}(u,T))$. Notice that if T has k edges then $d_{i,k}(u,T)$ is just the number of branches at u having exactly i edges.

Let G be a tree with k edges, and let U be the set of cut points of G. Let H be a tree and let V be the set of cut points of H. Denote by $G||H$ the following condition: For every $v \in V$ $d_k(v,H)$ is a linear combination with non-negative integer coefficients of the vectors $d_k(u,G)$ $(u \in U)$.

ARS COMBINATORIA, VOL. 12 (1981), pp. 117-121

Theorem 2 in [1] states that if G and H are trees and $e(G) > 1$ then $G|H \to G||H$. The converse is not true in general and some simple counter examples are given in [1].

In our notation, Question 3 in [1] is whether the following theorem is true:

THEOREM 1. *Let G be a tree with k edges, $k \geq 1$, and suppose that for every tree T with k edges*

$$G||T \to G|T \quad (\text{i.e. } G||T \to T \text{ is isomorphic to } G).$$

Then for every tree H

$$G||H \to G|H.$$

We shall prove this Theorem below. We note that its validity establishes Theorems 3,4,5 and 6 of [1] as immediate consequences of Lemmas 1-8 of [1], and using it we can prove many more theorems of the same kind.

Proof of Theorem 1. Let G be a tree satisfying the hypothesis of the theorem. Assume $k > 1$ since otherwise there is nothing to prove. An easy observation, stated as Observation 1 in [1], states that if $G||H$ then the number of edges of H is $m \cdot k$ for some $m \geq 1$. We prove the assertion of the theorem by induction on m. For $m = 1$ it holds by the hypothesis. Assume it holds for every $m' < m$, let H be a tree with $m \cdot k$ edges, $m > 1$, and suppose $G||H$.

If there is a cut point u of H, and a branch B at u with $m_1 \cdot k$ edges for some $0 < m_1 < m$, put $H_1 = B$ and let H_2 be the union of the remaining branches at u. Clearly H_1 and H_2 are trees. It is easily seen that every cut point v of H_1 is a cut point of H and $d_k(v,H_1) = d_k(v,H)$. Indeed this holds since $\mathcal{B}(v,H_1)\backslash \{B(v,u,H_1)\} = \mathcal{B}(v,H)\backslash\{B(v,u,H)\}$ and since

$$e(B(v,u,H) - e(B(v,u,H_1)) = e(H_2) = (m-m_1)\cdot k$$

implies $e(B(v,u,H)) \equiv e(B(v,u,H_1)) \pmod{k}$.

Similarly, every cut point $v \neq u$ of H_2 is a cut point of H and $d_k(v,H_2) = d_k(v,H)$. If u itself is a cut point of H_2, then clearly $d_k(u,H_2) = d_k(u,H)$. (Note the absence of the coordinate $d_{0,k}(u,T)$ from the vector $d_k(u,T)$!). Therefore $G||H_1$ and $G||H_2$, and by the induction hypothesis $G|H_1$ and $G|H_2$, which implies $G|H$, as needed.

Thus we may assume that the number of edges of any branch of H is not divisible by k. Let U denote the set of all cut points of G.

By hypothesis, for every cut point v of H, there are non-negative integers $x(v,u)$ $(u \in U)$, such that

$$d_k(v,H) = \Sigma\{x(v,u)\cdot d_k(u,G): \ u \in U\}.$$

We consider two cases.

<u>CASE 1</u>. There is a cut point v of H, such that

$$\Sigma\{x(v,u): \ u \in U\} > 1.$$

In this case, let $u_0 \in U$ satisfy $x(v,u_0) \geq 1$. Clearly there is an injective map

$$f: \ \mathcal{B}(u_0,G) \to \mathcal{B}(v,H)$$

such that

$$e(B) \equiv e(f(B)) \pmod k \quad \text{for all } B \in \mathcal{B}(u_0,G).$$

Put $H_1 = \cup\{f(B): \ B \in \mathcal{B}(u_0,G)\}$ and $H_2 = \cup\{B \in \mathcal{B}(v,H): \ B \notin f(\mathcal{B}(u_0,G))\}$. Clearly H_1 and H_2 are edge-disjoint trees and $E(H) = E(H_1) \cup E(H_2)$. Since f is not surjective, both H_1 and H_2 have fewer edges than H. Clearly $\Sigma\{e(B): \ B \in \mathcal{B}(u_0,G)\} = e(G) = k$ and thus $e(H_1) = \Sigma\{e(f(B)): B \in \mathcal{B}(u_0,G)\} \equiv 0 \pmod k$ and $e(H_2) = e(H) - e(H_1) \equiv 0 \pmod k$. As above it is easily checked, that for every cut point $w \neq v$ of H_1 $d_k(w,H_1) = d_k(w,H)$, and that for every cut point $q \neq v$ of H_2 $d_k(q,H_2) = d_k(q,H)$. In addition $d_k(v,H_1) = d_k(u_0,G)$ and $d_k(v,H_2) = \Sigma\{x(v,u).d_k(u,G): \ u \in U\setminus\{u_0\}\} + (x(v,u_0)-1).d_k(u_0,G)$.

Therefore $G||H_1$ and $G||H_2$. By the induction hypothesis $G|H_1$ and $G|H_2$ and thus $G|H$. This completes the proof of Case 1.

<u>CASE 2</u>. For every cut point v of H, $d_k(v,H) = d_k(u,G)$ for some

$u \in U$.

Recall that we assume that for no branch B of H $e(B) \equiv 0 \pmod k$. We shall prove that under this assumption Case 2 is impossible. Indeed suppose we are in Case 2 and let v be a cut point of H. By assumption there is a cut point u of G and a bijection $f: \mathcal{B}(u,G) \to \mathcal{B}(v,H)$ such that $e(B) \equiv e(f(B)) \pmod k$ for all $B \in \mathcal{B}(u,G)$. If $e(B) \le k$ for all $B \in \mathcal{B}(v,H)$, then $e(B) = e(f(B))$ for all $B \in \mathcal{B}(u,G)$, and thus

$$k = e(G) = \Sigma\{e(B): \; B \in \mathcal{B}(u,G)\} = \Sigma\{e(f(B)): \; B \in \mathcal{B}(u,G)\} = e(H) = m.k,$$

contradicting the fact that $m > 1$. Thus, for every cut point v of H, there is a branch $B \in \mathcal{B}(v,H)$, such that

(1) $\qquad e(B) > k.$

Let B be a branch of H for which $e(B)$ is minimal among all branches of H having more than k edges. Assume that B is a branch at v and let w be the only vertex of B adjacent to v. Clearly w is a cut point of H (since $k > 0$), and for every $C \in \mathcal{B}(w,H)\setminus\{B(w,v,H)\}$ $e(C) < e(B)$. This, together with the minimality of B, implies

(2) $\qquad e(C) \le k$

for all $C \in \mathcal{B}(w,H)\setminus\{B(w,v,H)\}$.

By assumption there is a cut point z of G and a bijection $g: \; \mathcal{B}(w,H) \to \mathcal{B}(z,G)$ such that for every $B \in \mathcal{B}(w,H)$, $e(B) \equiv e(g(B)) \pmod k$. This and (2) imply that

(3) $\qquad e(C) = e(g(C))$

for all $C \in \mathcal{B}(w,H)\setminus\{B(w,v,H)\}$. (1), (2) and (3) imply
$k < e(B) = 1 + \Sigma\{e(C): \; C \in \mathcal{B}(w,H)\setminus\{B(w,v,H)\}\} =$
$1 + \Sigma\{e(g(C)): \; C \in \mathcal{B}(w,H)\setminus\{B(w,v,H)\}\} =$
$1 + \Sigma\{e(B): \; B \in \mathcal{B}(z,G)\} - e(g(B(w,v,H))) \le 1 + k - 1 = k$, which is impossible.

Thus Case 2 is impossible and the validity of Theorem 1 is established. □

As we remarked, using Theorem 1 we can prove many results of the same kind as Theorems 3,4,5 and 6 of [1]. Theorem 2 below is one such result. We note that it implies Theorems 3,4, and 5 of [1] as special cases, and it is certainly more general than these three theorems.

THEOREM 2. *For* $r \geq 2$ *and* $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$, *let* $T(n_1,n_2,\ldots,n_r)$ *denote the tree consisting of* r *paths* $P_1,P_2,\ldots,P_r$ *meeting at one endpoint where* $e(P_i) = n_i$ *for* $1 \leq i \leq r$.

If $r = 2$ *or if* $r > 2$ *and* $n_1 \leq n_2 + \ldots + n_r$, *then for every tree* H

$$T(n_1,n_2,\ldots,n_r)|H \leftrightarrow T(n_1,n_2,\ldots,n_r)||H.$$

We omit the proof of Theorem 2, since it follows quite easily from Theorem 1.

References

[1] Y. Caro and J. Schonheim, *Decomposition of trees into isomorphic subtrees*. Ars Combinatoria 9(1980), 119-130.

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